On the State Complexity of Unary k-Entry DFAs

Summary & Key Examples

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What is a Unary k-Entry DFA?

- **Standard DFA:** One initial state (q_1).
- k-**Entry DFA:** A set of initial states $I = \{i_1, \ldots, i_k\}$ ($k \geq 1$).
- **Nondeterminism?** Only at the start, the transitions remain deterministic.
- **The Language:**

$$L(M) = \bigcup_{i \in I} L(M_i)$$

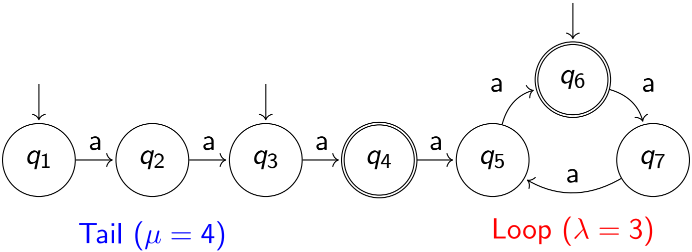
It accepts the *union* of languages starting from each initial point.

Why Unary?

Unary automata ($\Sigma = \{a\}$) have a distinct "Frying Pan" structure:

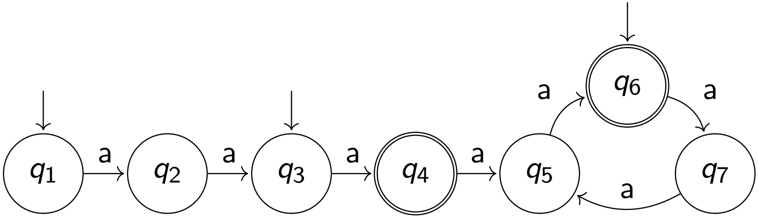
- A **Tail** (pre-period) of length μ.
- A **Loop** (period) of length λ.

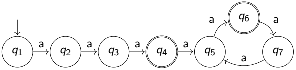
Visualizing the Structure



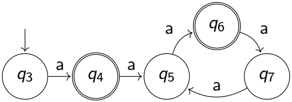
Here, $n = 7$. We have multiple entry points into a deterministic structure.

Decomposition

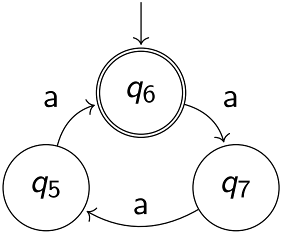




Initial State: q_1



Initial State: q_3

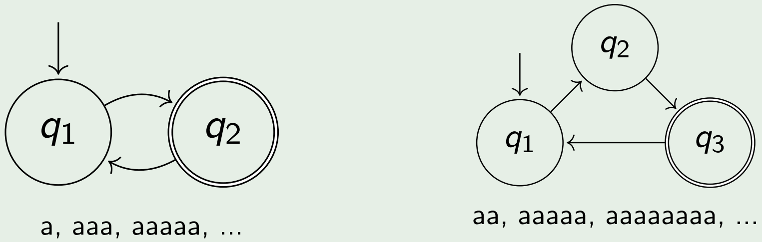


Initial State: q_6

Problem 1: The Multi-Union

Since $L(M) = \bigcup L(M_i)$, finding the minimal DFA is equivalent to finding the union of m unary languages.

Example



Resulting Complexity: The equivalent minimal DFA must track *both* periods.

$$\text{New Loop Size} = \text{lcm}(\lambda_1, \lambda_2) = \text{lcm}(2, 3) = 6$$

The general bound is:

$$\text{Size} \approx \text{lcm}(\lambda_1, \ldots, \lambda_m) + \max(\mu_1, \ldots, \mu_m)$$

Problem 2: Conversion of ${}_k$DFA to DFA

What if the graph is connected (one component)?

- **Intuition:** All initial states feed into the *same* cycle.
- Starting later in the tail just shifts the language: $L(q_j) = a^{-j} L(q_1)$.
- The period λ does not change

Upper Bound

For a weakly connected n-state ${}_k$DFA, the equivalent minimal DFA has **at most** n **states**.

Why? We don't need to multiply loop lengths. We just overlay the shifted languages on the same structure.

Problem 2: Conversion of ${}_k$DFA to DFA

If the ${}_k$DFA has multiple disconnected components:

- We can pick initial states in totally different loops.
- This triggers the LCM explosion we saw earlier.

Upper Bound

For components with sizes (λ_i, μ_i):

$$\text{States} \leq \text{lcm}(\lambda_1, \ldots, \lambda_m) + \max(\mu_1, \ldots, \mu_m)$$

This is related to the **Landau Function** $g(n)$, the maximal order of an element in the symmetric group S_n.

$$g(n) = \max\{\text{lcm}(x_1, \ldots, x_m) \mid x_1 + \cdots + x_m = n\} = e^{\sqrt{n \ln n}(1+o(1))}$$

Problem 3:Magic Numbers

Can we always build a minimal DFA of size exactly α from an n-state k-entry DFA?

Weakly Connected:

- Yes.
- We can achieve any size $\alpha \in [1, n]$.
- **Strategy:** Pick initial states to "skip" parts of the tail.

General Case:

- No.
- Gaps appear near the Landau function limit.
- **Example:** We cannot reach $g(n) - 1$ states for $n \geq 7$.

Other activities

- PAZ1a - teaching assistant 4 hours/week
- Informaticka kapustnica

Thank You!

Any Questions?