Operational state complexity of union, intersection and concatenation over unary automata with half of the states final

Student Scientific Conference 2024

Author: Petra Plšková

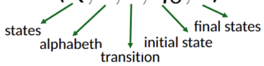
Supervisor: RNDr. Juraj Šebej, PhD.

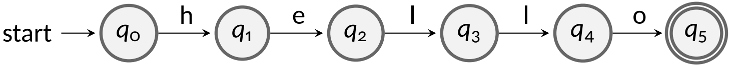
PRÍRODOVEDECKÁ FAKULTA UNIVERZITY P. J. ŠAFÁRIKA V KOŠICIACH • 1963 •

Automata theory

Definition of deterministic finite automata (DFA):

$$A = (Q, \Sigma, \delta, q_0, F)$$



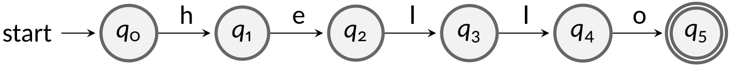


Automata theory

Definition of deterministic finite automata (DFA):

$$A = (Q, \Sigma, \delta, q_0, F)$$

states, alphabeth, transition, initial state, final states



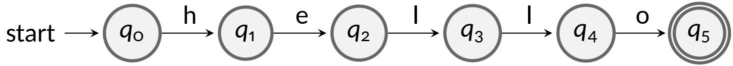
automata $\iff$ regular languages $\iff$ RegEx

Automata theory

Definition of deterministic finite automata (DFA):

$$A = (Q, \Sigma, \delta, q_0, F)$$

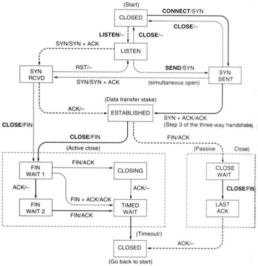
states, alphabeth, transition, initial state, final states



automata $\Longleftrightarrow$ regular languages $\Longleftrightarrow$ RegEx

"Real word" usage of finite automata:

- text processing (e.g. tokenization, morphological analysis, part-of-speech tagging in NLP)
- network protocols (e.g. RFC 793 for TCP protocol)
- compilers (if | else | while | for | return)
- hardware design (e.g. network cards)
- DNA sequence analysis



State complexity

automata $\Longleftrightarrow$ regular languages

State complexity characterizes the **cost**, in terms of states, of some basic operations (union, intersection, concatenation, etc.) on regular languages.

State complexity

automata $\Longleftrightarrow$ regular languages

State complexity characterizes the **cost**, in terms of states, of some basic operations (union, intersection, concatenation, etc.) on regular languages.

Definition (State complexity of regular languages)

The *deterministic state complexity of a regular language* L, $sc(L)$, is the number of states in the minimal DFA for L.

Definition (Operational state complexity)

The deterministic state complexity of a k-ary operation $\Box$ over a subclass $\mathcal{C}$ of DFAs is a function $sc : \mathbb{N}^k \to \mathbb{N}$ defined by

$$sc^{\mathcal{C}}_{\Box}(n_1, n_2, \ldots, n_k) = \max(sc(\Box(L(A_1), \ldots, L(A_k))) \mid A_i \in \mathcal{C} \text{ having } n_i \text{ states}, \; i \in \{1, \ldots, k\})$$

State complexity - finding minimal automaton

1. construct an NFA for the language $L(A)\Box L(B)$,
2. determinize the NFA to obtain a DFA,
3. minimize the DFA.

Motivation

Why are we studying *UNARY* automata with *HALF* of the states final?

Motivation

1. HALF OF THE STATES FINAL - equivalence between AFA and DFA with of the states

Theorem (A.FELLAH; JÜRGENSEN; YU, 1990)

Language L is accepted by an n-state AFA if and only if L^R is accepted by a 2^n-state DFA with half of the states final.

Motivation

1. HALF OF THE STATES FINAL - equivalence between AFA and DFA with of the states

Theorem (A.FELLAH; JÜRGENSEN; YU, 1990)

Language L is accepted by an n-state AFA if and only if L^R is accepted by a 2^n-state DFA with half of the states final.

2. UNARY AUTOMATA
   - The **state complexity** may be significantly **smaller** in the unary case compared to that of at least a two-letter alphabet."

Motivation

1. HALF OF THE STATES FINAL - equivalence between AFA and DFA with of the states

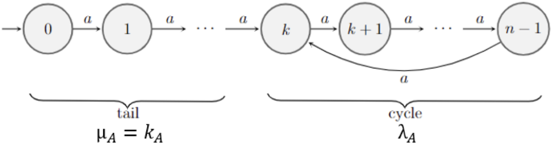
Theorem (A.FELLAH; JÜRGENSEN; YU, 1990)

Language L is accepted by an n-state AFA if and only if L^R is accepted by a 2^n-state DFA with half of the states final.

2. UNARY AUTOMATA
   - The **state complexity** may be significantly **smaller** in the unary case compared to that of at least a two-letter alphabet."
3. WHOLE SCALE OF EVEN NUMBER OF THE STATES
   - some patterns may be observed by having any n-state automata, $n \in \mathbb{N}$ even, not just powers of two
   - operation implemented in the program works faster on smaller automata

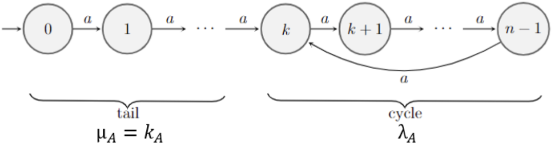
Unary deterministic finite automata

Considering complete unary DFAs with no unreachable states, the following transition diagram represents any such an unary DFA, omitting the finality of the states.



Unary deterministic finite automata

Considering complete unary DFAs with no unreachable states, the following transition diagram represents any such an unary DFA, omitting the finality of the states.



Nicaud's notation: $A = (n, k, F)$, where F is the set if final states

We slightly modify it to $A = (n_A, k_A, f_A)$

- $n_A \in \mathbb{N}$ is number of the states
- $k_A \in \mathbb{N}, k_A \leq n_A$ is length of the tail
- $f_A = b_0^A b_1^A \ldots b_{n-1}^A \in \{0,1\}^n$

$$b_i^A = \begin{cases} 1 & \text{if } i \in F \\ 0 & \text{if } i \notin F \end{cases}$$

Minimal unary automata

Necessary and sufficient condition for unary DFA minimality (PIGHIZZINI; SHALLIT, 2002)

An unary DFA $A = (Q, \{a\}, \delta, q_0, F)$ of size (μ, λ) is minimal if and only if both the following conditions are satisfied:

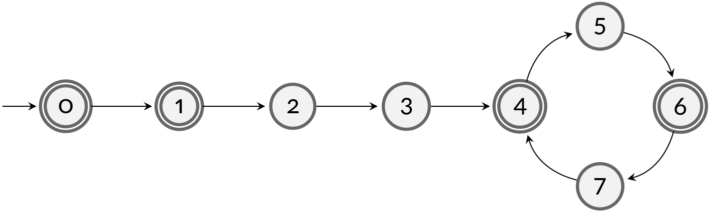
1. for any maximal proper divisor d of λ, there exists an integer $h \in \{0, ..., \lambda\}$ such that $p_h \in F$ if and only if $p_{(h+d)(mod \lambda)} \notin F$
2. $q_{\mu-1} \in F$ if and only if $p_{\lambda-1} \notin F$

Here $Q = \{q_0, q_1, ..., q_{\mu-1}, p_0, p_1, ..., p_{\lambda-1}\}$ and $\delta(q_i, a) = q_{i+1}$ for $i \in \{0, ..., q_{\mu-2}\}$, $\delta(q_{\mu-1}, a) = p_0$, $\delta(q_j, a) = q_{j+1}$ for $j \in \{0, ..., q_{\lambda-2}\}$, $\delta(p_{\lambda-1}, a) = p_0$.

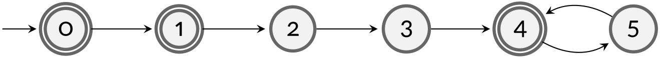
Conclusion: There is no need to use Hopcroft's minimization process

Minimal unary automata

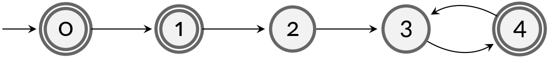
Example Reduction process for $A = (8, 4, 11001010)$



Can be reduced to:



Can be reduced to:



$A_{min} = (5, 3, 11001)$

Intersection and Union

State complexity of intersection/union over unary DFA (PIGHIZZINI; SHALLIT, 2002)

Let A be an unary DFA with the tail of length μ_A and the cycle of length λ_A and B be an unary DFA with the tail of length μ_B and the cycle of length λ_B. Then languages $L(A) \cup L(B)$ and $L(A) \cap L(B)$ are accepted by a DFA with the tail of length $\max(\mu_A, \mu_B)$ and the cycle of length (λ_A, λ_B)

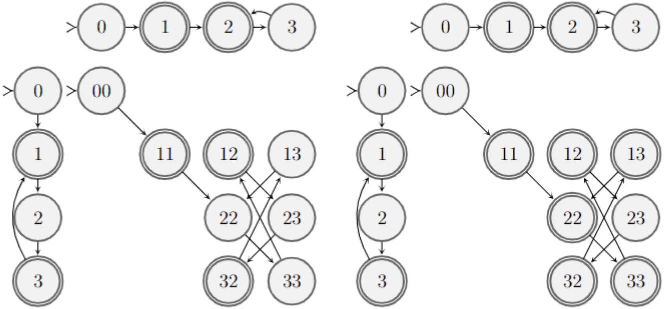
$$sc_\cap(m,n) = sc_\cup(m,n) = \max_{\lambda_A \in \{1,\dots,n\}, \lambda_B \in \{1,\dots,m\}} (\max\{n-\lambda_A, m-\lambda_B\} + nsn(\lambda_A, \lambda_B))$$

$$L(A) \cup L(B) = (L(A)^C \cap L(B)^C)^C$$

The last equality holds even for class of unary automata with half the states final

Intersection and Union

Example $A = (4, 2, 0110)$ and $B = (4, 1, 0101)$

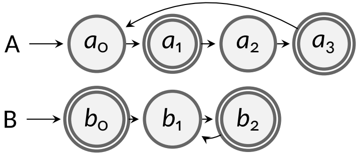


$f_{A \cap B} = (01(10)^3$ bitwise AND $0(101)^2 1)$
$f_{A \cup B} = (01(10)^3$ bitwise OR $0(101)^2 1)$

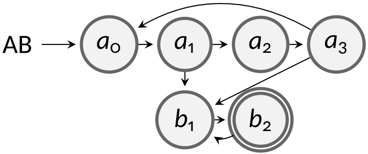
Concatenation

State complexity of concatenation over unary DFA (PIGHIZZINI; SHALLIT, 2002)

NLet A be an unary DFA with the tail of length μ_A and the cycle of length λ_A and B be an unary DFA with the tail of length μ_B and the cycle of length λ_B. Then the language $L(A) \cdot L(B)$ is accepted by a DFA with the tail of length $\mu_A + \mu_B + (\lambda_A, \lambda_B) - 1$ and the cycle of length (λ_A, λ_B).



Input DFAs A and B



Output NFA AB

Program

Class `unary_automaton` initializes itself to (n, k, F), its objects are inputs/output of the following functions

- `reduct` - outputs minimal DFA A_{min}
- `intersection` - outputs DFA $A \cap B$
- `union` - outputs DFA $A \cup B$
- `concatenation` - outputs DFA AB
- `square` - outputs DFA A^2
- `power` - outputs DFA A^k for given k
- `plus` - outputs DFA A^+
- `star` - outputs DFA A^*
- `complement` - outputs DFA A^C
- `minus` - outputs DFA $A - B$

Intersection results for m,n up to 10

	2	4	6	8	10
2	3	7	11	15	19
4	7	13	21	29	37
6	11	21	31	43	46
8	15	29	43	57	73
10	19	37	46	73	91

Table: State complexity intersection over unary DFAs with half of the states final

Hypothesis: $sc_{\cap}^{\mathcal{C}}(m,n) = sc_{\cap}^{\mathcal{C}_{1/2}}(m,n)$

Intersection results for m,n up to 10

	2	4	6	8	10
2	3	7	11	15	19
4	7	13	21	29	37
6	11	21	31	43	46
8	15	29	43	57	73
10	19	37	46	73	91

Table: State complexity intersection over unary DFAs with half of the states final

Hypothesis: $sc_{\cap}^{\mathcal{C}}(m,n) = sc_{\cap}^{\mathcal{C}_{1/2}}(m,n)$

	2	4	6	8	10
2	1	4	12	40	140
4	4	8	24	80	280
6	12	24	108	360	840
8	40	80	360	1280	4480
10	140	280	840	4480	17500

Table: Number of witnesses for the state complexity intersection over unary DFAs with half of the states final

Intersection results for m,n up to 10

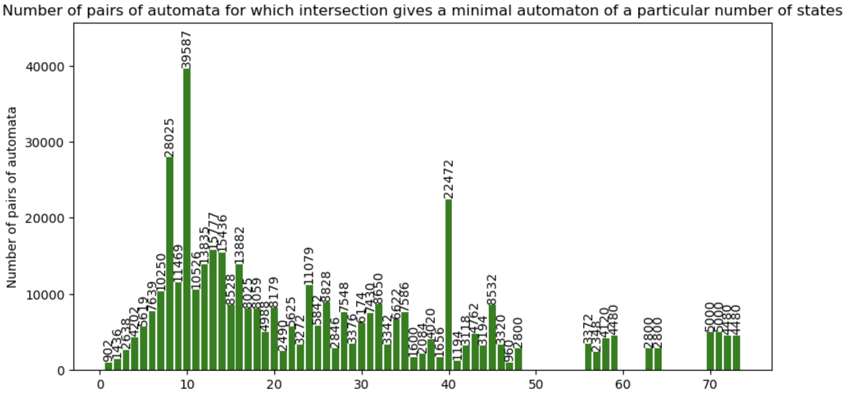


Figure: State complexities distributions for $m = 8$, $n = 10$

Remark: There are magic numbers

Intersection results for m,n up to 10

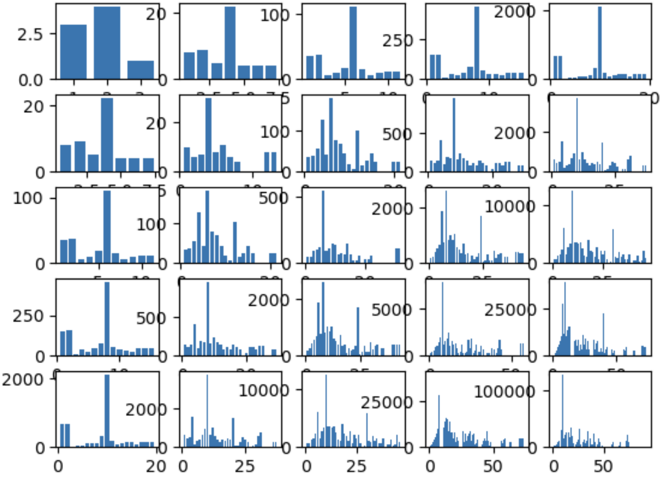


Figure: State complexities distributions for m,n up to 10

Intersection witnesses

m	n	$A = (m, k_A, f_A)$	$B = (n, k_B, f_B)$
2	2	(2,0,10)	(2,1,01)
2	4	(2,0,10)	(4,1,1100)
2	6	(2,0,10)	(6,1,111000)
2	8	(2,0,10)	(8,1,11110000)
2	10	(2,0,10)	(10,1,1111100000)
4	4	(4,0,1100)	(4,1,1100)
4	6	(4,0,1100)	(6,1,111000)
4	8	(4,0,1100)	(8,1,11110000)
4	10	(4,0,1100)	(10,1,1111100000)
6	6	(6,0,111000)	(6,1,111000)
6	8	(6,0,111000)	(8,1,11110000)
6	10	(6, 1 ,111000)	(10,1,1111100000)
8	8	(8,0,11110000)	(8,1,11110000)
8	10	(8,0,11110000)	(10,1,1111100000)
10	10	(10,0,1111100000)	(10,1,1111100000)

Concatenation results for m,n up to 10

	2	4	6	8	10
2	3	6	8	10	12
4	6	8	12	15	18
6	8	12	15	20	24
8	10	15	20	24	30
10	12	18	24	30	35

Table: State complexity of concatenation over unary DFAs with half of the states final

Hypothesis: $sc_{\circ}^{\mathcal{C}_{1/2}}(m,n) = \begin{cases} \frac{mn}{4} + \frac{m+n}{2} + 1 = (\frac{m}{2}+1)(\frac{n}{2}+1) & \text{if } m \neq n \\ \frac{n^2}{4} + n & \text{if } m = n \end{cases}$

Concatenation results for m,n up to 10

	2	4	6	8	10
2	3	6	8	10	12
4	6	8	12	15	18
6	8	12	15	20	24
8	10	15	20	24	30
10	12	18	24	30	35

Table: State complexity of concatenation over unary DFAs with half of the states final

Hypothesis: $sc_{\circ}^{\mathcal{C}_{1/2}}(m,n) = \begin{cases} \frac{mn}{4} + \frac{m+n}{2} + 1 = (\frac{m}{2}+1)(\frac{n}{2}+1) & \text{if } m \neq n \\ \frac{n^2}{4} + n & \text{if } m = n \end{cases}$

	2	4	6	8	10
2	2	1	2	3	4
4	1	2	3	3	2
6	2	3	2	2	1
8	3	3	2	2	2
10	4	2	1	2	3

Concatenation results for m,n up to 10

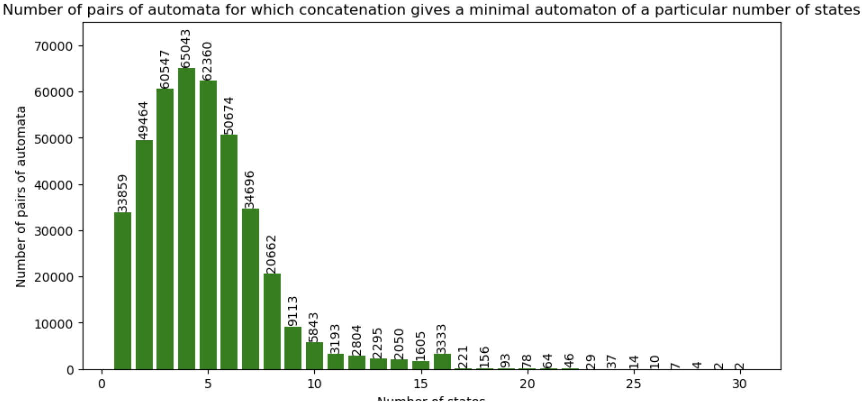


Figure: State complexities distributions for $m = 8$, $n = 10$

Remark: There are NO magic numbers

Concatenation results for m,n up to 10

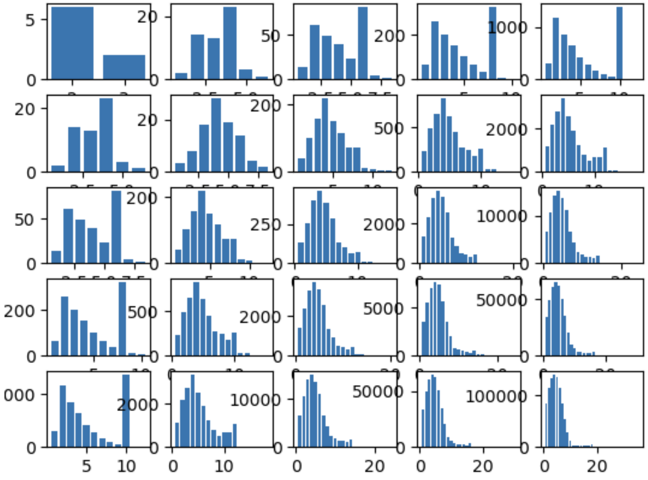


Figure: State complexities distributions for m,n up to 10

Concatenation witnesses

conditions	examples of possible witnesses
$m = 2$ and $n > 2$ k odd, $k \neq n-1$	$A = (2, 0, 01)$ $B = (n, k, (10)^{\frac{n}{2}})$ $AB = (2n - k + 1, 2n - k - 1, (01)^{\frac{n}{2}+1}1^{n-k-1})$ $AB_{min} = (n+2, n+1, (01)^{\frac{n}{2}+1})$
$n = m$ $m, n > 4$	$A = (m, \frac{m}{2} - 1, 1^{\frac{m}{2}-1}010^{\frac{m}{2}-1})$ $B = (m, \frac{m}{2}, 1^{\frac{m}{2}-1}0^{\frac{m}{2}}1)$ $AB = (\frac{m^2}{2} + \frac{3m}{2}, \frac{m^2}{4} + m - 1, 1^{\frac{m^2}{4}+\frac{m}{2}-2}01^{\frac{m}{2}-1}01^{\frac{m}{2}+1})$ $AB_{min} = (\frac{m^2}{4} + m, \frac{m^2}{4} + m - 1, 1^{\frac{m^2}{4}+\frac{m}{2}-2}01^{\frac{m}{2}-1}01)$
$n = m$ $m, n > 4$	$A = (m, \frac{m}{2} - 1, 1^{\frac{m}{2}}0^{\frac{m}{2}})$ $B = (m, \frac{m}{2} - 2, 1^{\frac{m}{2}-2}000110^{\frac{m}{2}-3})$ $AB = (\frac{m^2}{4} + \frac{5m}{2} - 1, \frac{m^2}{4} + 2m - 2, 1^{\frac{m^2}{4}-6}01^{\frac{m}{2}+1}01^{\frac{m}{2}+1}01^{\frac{3m}{2}})$ $AB_{min} = (\frac{m^2}{4} + m, \frac{m^2}{4} + m - 1, 1^{\frac{m^2}{4}-6}01^{\frac{m}{2}+1}01^{\frac{m}{2}+1}01)$

Concatenation witnesses

conditions	examples of possible witnesses
$n = m+2$	$A = (m, \frac{m}{2} - 1, 1^{\frac{m}{2}-1}010^{\frac{m}{2}-1})$
$m, n > 2$	$B = (m+2, \frac{m}{2}, 1^{\frac{m}{2}}010^{\frac{m}{2}})$
	$AB = (\frac{m^2}{4} + \frac{5m}{2} + 2, \frac{m^2}{4} + 2m + 1, 1^m01^{\frac{m^2}{4}+\frac{m}{2}-1}01^{m+1})$
	$AB_{min} = (\frac{m^2}{4} + \frac{3m}{2} + 2, \frac{m^2}{4} + \frac{3m}{2} + 1, 1^m01^{\frac{m^2}{4}+\frac{m}{2}-1}01)$
$n = m+4$	$A = (m, \frac{m}{2} - 1, 1^{\frac{m}{2}-1}010^{\frac{m}{2}-1})$
$m, n > 2$	$B = (m+4, \frac{m}{2} + 2, 1^{\frac{m}{2}}010^{\frac{m}{2}}10)$
	$AB = (\frac{m^2}{4} + \frac{5m}{2} + 4, \frac{m^2}{4} + 2m + 3, 1^m01^{\frac{m}{2}}01^{\frac{m^2}{4}+\frac{m}{2}-1}01^{\frac{m}{2}+2})$
	$AB_{min} = (\frac{m^2}{4} + 2m + 3, \frac{m^2}{4} + 2m + 2, 1^m01^{\frac{m}{2}}01^{\frac{m^2}{4}+\frac{m}{2}-1}01)$
$n = m+6$	$A = (m, \frac{m}{2} - 1, 1^{\frac{m}{2}-1}010^{\frac{m}{2}-1})$
$m, n > 2, 4 \mid m$	$B = (m+6, \frac{m}{2} + 3, 1^{\frac{m}{2}}0110^{\frac{m}{2}}100)$
	$AB = (\frac{m^2}{4} + 3m + 6, \frac{m^2}{4} + \frac{5m}{2} + 5, 1^{\frac{m^2}{8}+\frac{7m}{4}+1}01^{\frac{m}{2}}01^{\frac{m^2}{8}+\frac{m}{4}-1}01^{\frac{m}{2}+3})$
	$AB_{min} = (\frac{m^2}{4} + \frac{5m}{2} + 4, \frac{m^2}{4} + \frac{5m}{2} + 3, 1^{\frac{m^2}{8}+\frac{7m}{4}+1}01^{\frac{m}{2}}01^{\frac{m^2}{8}+\frac{m}{4}-1}01)$

State complexity of intersection

Theorem 3.1 (Intersection - lower bound, special case)

Let $A = (m, 0, 1^{\frac{m}{2}}0^{\frac{m}{2}})$, $B = (n, 1, 1^{\frac{n}{2}}0^{\frac{n}{2}})$ be unary deterministic finite automata where $m, n \in \mathbb{N}$ are even, $m \leq n$, $gcd(m, n-1) = 1$, $n > 2$. Then

$$sc(L(A) \cap L(B)) = mn - m + 1$$

State complexity of intersection

Theorem 3.1 (Intersection - lower bound, special case)

Let $A = (m, 0, 1^{\frac{m}{2}}0^{\frac{m}{2}})$, $B = (n, 1, 1^{\frac{n}{2}}0^{\frac{n}{2}})$ be unary deterministic finite automata where $m, n \in \mathbb{N}$ are even, $m \leq n$, $gcd(m, n-1) = 1$, $n > 2$. Then

$$sc(L(A) \cap L(B)) = mn - m + 1$$

Sketch of proof

$A \cap B = (lcm(m, n-1) + max(0, 1), max(0, 1), f_{A \cap B}) = (mn - m + 1, 1, f_{A \cap B})$, where $f_{A \cap B}$ is the result of the bitwise AND operation on the following words

$$(1^{\frac{m}{2}}0^{\frac{m}{2}})^{n-1}1$$

$$1(1^{\frac{n}{2}-1}0^{\frac{n}{2}})^{m}$$

We prove that $f_{A \cap B}$ won't minimize with regard to the "condition for unary DFA minimality". The second condition clearly holds and the proof for the first condition is done by contradiction.

State complexity of intersection

Theorem 3.5 (Intersection/union - state complexity)

Let $\mathcal{C}$ be a class of all unary DFA and $\mathcal{C}_{1/2}$ be a class of all unary DFA with half of the states final; $m, n \in \mathbb{N}$ even. Then

$$sc_{\cap}^{\mathcal{C}}(m,n) = sc_{\cap}^{\mathcal{C}_{1/2}}(m,n)$$

$$sc_{\cup}^{\mathcal{C}}(m,n) = sc_{\cup}^{\mathcal{C}_{1/2}}(m,n)$$

State complexity of intersection

Theorem 3.5 (Intersection/union - state complexity)

Let $\mathcal{C}$ be a class of all unary DFA and $\mathcal{C}_{1/2}$ be a class of all unary DFA with half of the states final; $m, n \in \mathbb{N}$ even. Then

$$sc_{\cap}^{\mathcal{C}}(m, n) = sc_{\cap}^{\mathcal{C}_{1/2}}(m, n)$$

$$sc_{\cup}^{\mathcal{C}}(m, n) = sc_{\cup}^{\mathcal{C}_{1/2}}(m, n)$$

Sketch of proof Given $k_A \geq k_B$, $\lambda_A = m - k_A$, $\lambda_B = n - k_B$, $Q(\lambda_A, \lambda_B) = \max(m - \lambda_A, n - \lambda_B) + (\lambda_A, \lambda_B)$ maximized, the have witnesses

$$A = (m, m - \lambda_A, 1^{\frac{m}{2} - \lceil \frac{\lambda_A}{2} \rceil} 0^{\frac{m}{2}} 1^{\lceil \frac{\lambda_A}{2} \rceil}), \qquad B = \begin{cases} (n, n - \lambda_B, 1^{\frac{n}{2} - 1} 0^{\frac{n}{2}} 1), & \text{if } j < \frac{n}{2} \text{ or } j = n \\ (n, n - \lambda_B, 0^{\frac{n}{2} - 1} 1^{\frac{n}{2}} 0), & \text{if } \frac{n}{2} \leq j < n \end{cases}$$

$$j = \begin{cases} k_B & \text{if } k_B = k_A \\ n & \text{if } k_B \neq k_A \text{ and } \lambda_B \mid (k_A - k_B) \\ k_B + ((k_A - k_B) \mod \lambda_B) & \text{if } k_B \neq k_A \text{ and } \lambda_B \nmid (k_A - k_B) \end{cases}$$

PRÍRODOVEDECKÁ FAKULTA
UNIVERZITA PAVLA JOZEFA ŠAFÁRIKA
V KOŠICIACH

State complexity of intersection

Krajňáková obtained the same results in her dissertation thesis (KRAJNAKOVA, 2020) with a different witnessing pair. One of her witnesses was not a minimal automaton; here, we present witnesses that are both minimal automata.

Theorem 3.8 (Intersection/union - state complexity AFA)

Let $m, n \in \mathbb{N}$. Considering the class of all unary alternating finite automata we get

$$asc_{\cap}(m,n) = acs_{\cup}(m,n) = m + n + 1$$

State complexity of intersection

Krajňáková obtained the same results in her dissertation thesis (KRAJNAKOVA, 2020) with a different witnessing pair. One of her witnesses was not a minimal automaton; here, we present witnesses that are both minimal automata.

Theorem 3.8 (Intersection/union - state complexity AFA)

Let $m, n \in \mathbb{N}$. Considering the class of all unary alternating finite automata we get

$$asc_{\cap}(m,n) = acs_{\cup}(m,n) = m+n+1$$

Sketch of proof $asc(L_1 \cap L_2) \leq m+n+1$ from (A.FELLAH; JÜRGENSEN; YU, 1990)

$$sc((L_1 \cap L_2)^R) = sc(L_1 \cap L_2) = 2^m 2^n - min(2^m, 2^n) + 1$$

$$asc(L_1 \cap L_2) \geq \lceil \log_2(2^m 2^n - min(2^m, 2^n) + 1) \rceil = m+n$$

$asc(L_1 \cap L_2) \geq m+n+1$ by contradiction: $(A \cap B)_{min}$ for $A = (2^m, 0, 1^{2^{m-1}} 0^{2^{m-1}})$ and $B = (2^n, 1, 1^{2^{n-1}} 0^{2^{n-1}})$ doesn't have exactly half of the states final.

State complexity of concatenation

IDEA: convert the task of concatenating languages into that of unioning languages, where the languages used for union are concatenations of simpler languages than the original

State complexity of concatenation

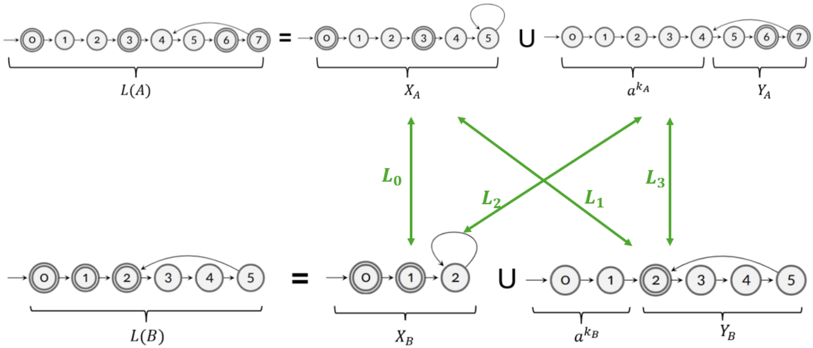
IDEA: convert the task of concatenating languages into that of unioning languages, where the languages used for union are concatenations of simpler languages than the original

Let $A = (n_A, k_A, f_A)$, $B = (n_B, k_B, f_B)$ be unary DFAs. It holds that

$$L(A) = X_A \cup a^{k_A} Y_A \text{ for } X_A = L(A) \cap \{a^i \mid i = 0, \dots, k_A - 1\}$$
$$Y_A = \{a^i \mid a^{i+k_B} \in L(A), i \in \mathbb{N}_0\}$$
$$L(B) = X_B \cup a^{k_B} Y_B \text{ for } X_B = L(B) \cap \{a^i \mid i = 0, \dots, k_B - 1\}$$
$$Y_B = \{a^i \mid a^{i+k_A} \in L(B), i \in \mathbb{N}_0\}$$

Therefore using distributive law we get $L(A) \cdot L(B) = L_0 \cup L_1 \cup L_2 \cup L_3$, where $L_0 = X_A X_B$, $L_1 = a^{k_B} X_A Y_B$, $L_2 = a^{k_A} X_B Y_A$, $L_3 = a^{k_A + k_B} Y_A Y_B$.

State complexity of concatenation



State complexity of concatenation

Theorem 3.10 (Concatenation - lower bound witnesses, the same length)

Let $A = (n, \frac{n}{2} - 1, 1^{\frac{n}{2}-1}010^{\frac{n}{2}-1})$, $B = (n, \frac{n}{2}, 1^{\frac{n}{2}-1}0^{\frac{n}{2}}1)$ be unary deterministic finite automata where $n \in \mathbb{N}$ even, $n \geq 4$. Then

$$sc(L(A) \cdot L(B)) = \frac{n^2}{4} + n$$

Sketch of proof

$$A_{L_0} = (n-2, n-3, 1^{n-3}0)$$

$$A_{L_1} = \left(\frac{3n}{2} - 1, n-1, 0^{n-1}1^{\frac{n}{2}-1}0\right)$$

$$A_{L_2} = \left(n+1, \frac{n}{2}, 0^{\frac{n}{2}}1^{\frac{n}{2}-1}00\right)$$

$$A_{L_3} = \left(\frac{n^2}{4} + n, \frac{n^2}{4} + n - 1, 0^{\frac{3n}{2}-1}(1^i0^{\frac{n}{2}-i})_{i=1}^{\frac{n}{2}-1}1\right)$$

State complexity of concatenation

Bitwise OR for the following expressions:

$$expr_0 = 1^{n-3}0^{\frac{n}{2}(\frac{n}{2}+1)}0^{\frac{n^2}{4}+2}$$

$$expr_1 = 0^{n-1}(1^{\frac{n}{2}-1}0)^{\frac{n}{2}+1}(1^{\frac{n}{2}-1}0)^{\frac{n}{2}}1$$

$$expr_2 = 0^{\frac{n}{2}}(1^{\frac{n}{2}-1}00)^{\frac{n}{2}}(1^{\frac{n}{2}-1}00)^{\frac{n}{2}-1}1^{\frac{n}{2}-1}01$$

$$expr_3 = 0^{\frac{3n}{2}-1}(1^i0^{\frac{n}{2}-i})_{i=1}^{\frac{n}{2}-1}1^{\frac{n}{2}(\frac{n}{2}+1)}$$

After minimization we get

$$AB_{min} = \left(\frac{n^2}{4}+n, \frac{n^2}{4}+n-1, 1^{\frac{n^2}{4}+\frac{n}{2}-2}01^{\frac{n}{2}-1}01\right).$$

Remark For $m = n+2$, $m = n+4$, $m = n+6$ is the proof of the hypothesis similar.

State complexity of concatenation

Lemma 3.12

Let $n \in \mathbb{N}, n \geq 4$. Let $\mathcal{C}$ be a class of unary deterministic finite automata with half of the states final such that

- both of their cycles contains only one final state,
- the greatest common divisor of lengths of their cycles is one.

Then $sc_{\circ}^{\mathcal{C}}(n, n) = \frac{n^2}{4} + n$. Moreover the bound is met if and only if $A = (n, \frac{n}{2} - 1, 1^{\frac{n}{2}-1}010^{\frac{n}{2}-1})$ and $B = (n, \frac{n}{2}, 1^{\frac{n}{2}-1}0^{\frac{n}{2}}1)$ or vice versa.

State complexity of concatenation

Theorems 3.15, 3.16, 3.17

1. Let $A = (n, \frac{n}{2} - 1, 1^{\frac{n}{2}-1}010^{\frac{n}{2}-1})$, $B = (n+2, \frac{n}{2}, 1^{\frac{n}{2}}010^{\frac{n}{2}})$ be unary deterministic finite automata where $n \in \mathbb{N}$ even, $n \geq 4$. Then

$$sc(L(A) \cdot L(B)) = \frac{n(n+2)}{4} + \frac{n+(n+2)}{2} + 1$$

2. Let $A = (n, \frac{n}{2} - 1, 1^{\frac{n}{2}-1}010^{\frac{n}{2}-1})$, $B = (n+4, \frac{n}{2}+2, 1^{\frac{n}{2}}010^{\frac{n}{2}}10)$ be unary deterministic finite automata where $n \in \mathbb{N}$ even, $n \geq 4$. Then

$$sc(L(A) \cdot L(B)) = \frac{n(n+4)}{4} + \frac{n+(n+4)}{2} + 1$$

3. Let $A = (n, \frac{n}{2} - 1, 1^{\frac{n}{2}-1}010^{\frac{n}{2}-1})$, $B = (n+6, \frac{n}{2}+3, 1^{\frac{n}{2}}0110^{\frac{n}{2}}100)$ be unary deterministic finite automata where $n \in \mathbb{N}$ even, $n \geq 4$, $4 \mid n$. Then

$$sc(L(A) \cdot L(B)) = \frac{n(n+6)}{4} + \frac{n+(n+6)}{2} + 1$$

State complexity of concatenation

Theorem 3.19 (Concatenation $m = n$, state complexity AFA)

Let $m, n \in \mathbb{N}$ and $m = n$. Considering the class of all unary alternating finite automata we get

$$m + n \leq asc_{\circ}(m, n) \leq m + n + 1$$

Sketch of proof

$$sc((L_1L_2)^R) = sc(L_1L_2) \geq (2^{m-1} + 1)(2^{n-1} + 1)$$

$$asc(L_1L_2) \geq \lceil \log_2((2^{m-1} + 1)(2^{n-1} + 1)) \rceil = m + n - 1$$

$asc(L_1L_2) \geq m + n$ by contradiction: $(AB)_{min}$ for $A = (2^m, 2^{m-1} - 1, 1^{2^{m-1}-1}010^{2^{m-1}-1})$ and $B = (2^n, 2^{n-1}, 1^{2^{n-1}-1}0^{2^{n-1}}1)$ doesn't have exactly half of the states final.

State complexity of square

Theorem 3.20 (State complexity of the square)

Let L be arbitrary languages accepted by a n-state DFA with half of the states final, $n \in \mathbb{N}$ even. Then

$$sc(L^2) = 2n - 1$$

Sketch of proof Upper bound: $2n - 1$
Witness: $A = (n, 0, 0^{\frac{n}{2}}1^{\frac{n}{2}})$
$A^2 = (2n, n, 0^n 1^{n-1} 0)$.
$A^2_{min} = (2n - 1, n - 1, 0^n 1^{n-1})$
upper bound = lower bound

Contribution

Recapitulation of my own contribution

- program implementation
  - basic operation over unary DFAs
  - statistics for m, n up to 10
  - hypothesis verification up to fixed m, n
- state complexity of **intersection** and **union**
  - both lower bound and its tightness for DFA with half of the states final
  - both lower bound and its tightness for DFA with any fixed number of final states
  - lower bound and its tighness for AFA
- state complexity of **concatenation**
  - lower bounds for $m = n$, $m = n + 2$, $m = n + 4$ and $m = n + 6$ if $4 \mid m$
  - tightness for $m = n$ if length of cycles are comprime and have one final state
  - lower bound for AFA if $m = n$
- state complexity of **square**
  - both lower bound and its tightness

Future work

Problems left open

- **intersection**
  - lower bound and tightness if one of the DFAs have exactly one final state
- **concatenation**
  - lower bounds for whole scale of m,n even (might be obtained by generalizing found witnesses)
  - tightness for the stated lower bounds
  - state complexity over AFA for the whole scale

Ďakujem za pozornosť!

Thank you for your attention!

Bibliography

A.FELLAH; JÜRGENSEN, H.; YU, S. Constructions for alternating finite automata. **International Journal of Computer Mathematics**, Taylor Francis, v. 35, n. 1-4, p. 117–132, 1990. 9, 10, 11, 35, 36

KRAJNAKOVA. **Finite automata and operational complexity**. Tese (Doutorado) — Comenius University in Bratislava, 2020. 35, 36

PIGHIZZINI, G.; SHALLIT, J. Unary language operations, state complexity and jacobsthal's function. **International Journal of Foundations of Computer Science**, v. 13, 02 2002. 14, 16, 18